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A Feynman Path Integral Representation for Elastic Wave Scattering by Anisotropic Weakly Perturbations

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Abstract We write a space-time Feynman Path Integral representation for scattered elastic wave fields from a weakly anisotropic non-homogeneity.

Keywords Feynman path integral · Anisotropic medium · Wave scattering · Seismic tomography in the Path Integral approach

1 Introduction

It has been an useful tool to study waves of various physical nature, the standard by now operatorial method of Feynman path integral [1, 2]. However the generalization of the path integral method to wave propagation in solids as well as liquids needs proposals for extension of the path integral technique to a full elastic-anisotropic medium, a subject of the utmost importance to Geophysics (see Ref. [3] for insights into the origin of the method of path summation for elastic waves). We aim in this note to contribute to this still unsolved problem of anisotropic wave propagations by considering a sort of approximate—first order perturbation Feynman path integral representation for wave scattering through a weakly anisotropic scattering potential [2].

2 The Anisotropic Path Integral Representation

Let us start our analysis by considering a wave equation associated to an "acoustic wave" scattered by an anisotropic center in R^N with a scalar space-time dependent damping

$$\frac{\partial^2}{\partial t^2}U^i(x,t) = C_{ijk\ell}(x)\frac{\partial^2}{\partial x^j\partial x^l}U^k(x,t) - \nu(x,t)\frac{\partial}{\partial t}U^i(x,t) + \delta\big(x-x'\big)\delta\big(t-t'\big)e_i \quad (1)$$

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where the time range is the whole real line $-\infty < t < \infty$, e_i are the standard canonical vectorial basis of R^N, $U^i(x,t)$ is the vector position of the wave field under scattering and $\gamma_{ijk\ell}(x)$ is a spatially-dependent anisotropic medium tensor carrying the anisotropic scattering center properties ($\gamma_{ijk\ell}(x) = \gamma_{jik\ell}(x) = \gamma_{ij\ell k}(x) = \gamma_{k\ell ij}(x)$)

$$C_{ijk\ell}(x) = \left[C_0^2(x)\delta_{ik}\delta_{j\ell} + \varepsilon\gamma_{ijk\ell}(x)\right]. \tag{2}$$

Note that $C_0^2(x)$ is the acoustic unperturbed wave velocity scalar function and $\gamma_{ijk\ell}(x)$ is a fourth-order "elasticity" tensor of the scattering anisotropic center and ε is a small parameter associated to the intensity of this wave field weakly perturbation. The positive definite function $\nu(x,t)$ describe our "losses" due to the existence of damping.

The Green function associated to our coefficients variable anisotropic wave equation (1) is supposed to have the operatorial-distributional rule above written as its definition ([2] and the convention $\gamma_{ajb\ell}v^j v^\ell \equiv \gamma_{ab}^{j\ell} v^j v^\ell$!) in quantum mechanical Dirac bra-ket notation[1]

$$\begin{aligned}
&(\mathcal{L})_{ik}^{-1}\big((x,t),\big(x',t'\big)\big) \\
&\stackrel{\text{def}}{\equiv} \left[\frac{\partial^2}{\partial t^2}\delta_{ab} + \nu(x,t)\frac{\partial}{\partial t}\delta_{ab} - C_0^2(x)\Delta\delta_{ab} - \varepsilon\gamma_{ab}^{j\ell}(x)\frac{\partial^2}{\partial x^j\partial x^\ell}\right]_{ik}^{-1}\big((x,t),\big(x',t'\big)\big) \\
&= \text{Real}\left(-i\int_0^\infty ds\big\langle(x,t,i)\big|\exp\big(is[\mathcal{L}]\big)\big|\big(x',t',k\big)\big\rangle\right).
\end{aligned} \tag{3}$$

Since ε is small and the "scalar acoustic tensor" $C_0^2(x)\delta_{ik}\delta_{j\ell}$ is fully diagonal, one can proceed as in Refs. [1, 2] and [3] and arrive at the full space-time wave field propagation generated by a source at the point x' and at the time t'. (See Appendix B for the calculational details.)

$$\begin{aligned}
&[\mathcal{L}]_{ik}^{-1}\big((x,t),\big(x',t'\big)\big) \\
&= \text{Real}\Bigg\{-i\int_0^\infty ds\int_{\substack{t(0)=t' \\ t(s)=t}} D^F\big[t(\sigma)\big]\int_{\substack{\vec r(0)=x' \\ \vec r(s)=x}} \overbrace{\Bigg(\prod_{0\le\sigma\le s} d\vec r(\sigma)\big(C_0\big(r(\sigma)\big)\big)^{-N/2}\Bigg)}^{d^{\text{cov}}[\vec r(\sigma)]} \\
&\quad\times\exp\left(\frac{i}{4}\int_0^s d\sigma\left(\frac{dt}{d\sigma}\right)^2\right)\exp\left(\frac{i}{4}\int_0^s d\sigma\,\nu^2\big(\vec r(\sigma),t(\sigma)\big)\right) \\
&\quad\times\exp\left(-\frac{1}{2}\int_0^s \nu\big(\vec r(\sigma),t(\sigma)\big)\frac{dt}{d\sigma}\right)\exp\left(-\frac{i}{4}\int_0^s d\sigma\frac{1}{C_0^2(\vec r(\sigma))}\left(\frac{d\vec r}{d\sigma}\right)^2\right)\Phi_{ik}\big[[\gamma],s\big]\Bigg\}
\end{aligned} \tag{4}$$

[1] We have used the L. Schwartz distributional integral representation for $\alpha > 0$:

$$-i\int_0^\infty e^{i\alpha z}\,dz \stackrel{D'(R)}{=} \overbrace{\frac{\pi}{i}\delta(\alpha)}^{=0} + \left(\frac{1}{\alpha}\right).$$

where we have introduced the Anisotropic Polarizations vector Factor directing the wave propagation along the anisotropics axis

$$\Phi_{ik}\big[[\gamma], s\big] \equiv \exp\left(T_{0\le\sigma\le s}\left\{\left[-i\varepsilon \int_0^s \overbrace{\left(\frac{\gamma_{pmqn}(\vec{r}(\sigma))}{(C_0(\vec{r}(\sigma)))^4}\right)\frac{dr^m(\sigma)}{d\sigma}\cdot\frac{dr^n(\sigma)}{d\sigma}}^{\text{matrix}(p,q)}\right]\right\}\right)_{ik}. \tag{5}$$

Note the use of an proper-time ordered—matrix indexes (p,q) on the above written anisotropic factor.

That is our general result. However, although a somewhat intricate object to be analytically evaluated [1, 2], in practical lossless geophysical/signal detection-engineering applications (with $\nu(x,t)\equiv 0$), one is always interested in the so called "short-time arrival signals".

At this WKB like approach for our path integrals,[2] Brownian like path trajectories $\{\vec{r}(\sigma)\}$ roughs/nondifferentiable curves are dominated by geometrical rays paths, i.e., those classical differentiable paths are dominant paths contributing to our path integral Feynman measures (including the "classical" weights!). We have thus the standard results ([1], Chap. 24)

$$\int_{t(0)=t'}^{t(s)=t} D^F\big[t(\sigma)\big] e^{(\frac{i}{4}\int_0^s(\frac{dt}{d\sigma})^2)} = \frac{e^{\frac{i}{4s}(t-t')^2}}{\sqrt{4\pi i s}}, \tag{6}$$

$$\int_{\vec{r}(0)=x'}^{\vec{r}(s)=x} d^{\text{cov}}\big[\vec{r}(\sigma)\big] e^{-\frac{i}{4}\int_0^s d\sigma (\frac{d\vec{r}}{d\sigma})^2 \frac{1}{C_0^2(\vec{r}(\sigma))}} = \frac{(C_0^{-N}(x)C_0^{-N}(x'))e^{\frac{i}{4s}\Delta^2(x,x')}}{(\sqrt{4\pi i s})^N} \tag{7}$$

and

$$(\Phi_{ik}-\delta_{ik}) \sim -i\varepsilon\left[\int_0^s d\sigma \left(\frac{\gamma_{imkn}(\vec{R}^{CL}(\sigma))}{C_0^4(\vec{R}^{CL}(\sigma))}\right)\dot{R}^{CL,m}(\sigma)\dot{R}^{CL,n}(\sigma)\right] \tag{8}$$

where the geometrical ray path satisfies the classical ODE's rays equation of classical optics

$$\frac{\delta}{\delta\vec{r}(\bar{\sigma})}\left\{\int_0^s d\sigma\left(\frac{d\vec{r}(\sigma)}{d\sigma}\right)^2 \frac{1}{C_0^2(\vec{r}(\sigma))}\right\}\Bigg|_{\vec{r}(\bar{\sigma})=\vec{R}^{CL}(\bar{\sigma})} \equiv 0 \tag{9}$$

and the geodesic length connecting the points of observation x and x' is given for a small deviation condition by the quadratic object

$$\Delta^2(x,x') = -\frac{1}{2}\left(\frac{1}{C_0^2(\frac{x+x'}{2})}(x-x')^2\right) + 0\big((x-x')^4\big). \tag{10}$$

It is worth point out that the anisotropic polarization factor (8) still depends on the wave field proper-time s. However in this "short-time arrivals" regime, one can expect that the

[2] This WKB's short-time arrival may be thought as the proper-time vanishing limit $S\to 0^+$ on the path integral weight:

$$\exp\left(-\frac{i}{4S}\int_0^1 d\gamma \frac{1}{C^2(r(\gamma))}\left(\frac{dr}{d\gamma}\right)^2\right).$$

proper-time integration can be considered as an average and thus can be factorized in our general (4) as a first significant leading approximation for the proper-time integration[3]

$$[\mathcal{L}]^{-1}_{ik}\big((x,t),\big(x',t'\big)\big) \simeq \text{Real}\left\{-i\left[\int_0^\infty ds(6)\times(7)\right]\times\left[\int_0^\infty ds(8)\right]\right\}. \tag{11}$$

In this effective case (11), one can easily disregard the "non-physical" advanced piece of the solution as we show in the simple example of a constant velocity medium in R^3, procedure to be done by hand, since there is an causal $i\varepsilon$ prescription already built in. See footnote 3.

$$\begin{aligned}
&\text{Real}\left(-i\int_0^\infty ds\,\frac{e^{\frac{i}{4}(t-t')^2/S}}{\sqrt{4\pi i s}}\times\frac{e^{\frac{i}{4s}[(t-t')^2-C_0^{-2}(x-x')^2]}}{(\sqrt{4\pi i s})^3}\right)\\
&=\text{Real}\left[-i\overbrace{\left(\frac{1}{\pm\frac{(1+i)}{\sqrt{2}}}\right)\left(\frac{1}{\pm\frac{(1-i)}{\sqrt{2}}}\right)}^{=+1}\times\int_0^\infty\frac{ds}{s^2}\exp\left[\frac{i}{4s}|t-t'|^2-C_0^{-2}|x-x'|^2\right]\right]\\
&=\frac{1}{2}\left(\frac{\delta(C_0|t-t'|-|x-x'|)}{C_0|x-x'|}\right)+\frac{1}{2}\left(\frac{\delta(C_0|t-t'|+|x-x'|)}{C_0|x-x'|}\right)\\
&=\frac{1}{2}\left(\frac{(\delta(C_0|t-t'|-|x-x'|)}{C_0|x-x'|}\right).
\end{aligned}\tag{12}$$

Finally, let us remark that (4)–(5) present interesting prospects for implementing numerical path-summation methods as exposed fully in Refs. [4–6], see also Appendix C for a toy-model of ours to understand the principles of path-integral tomography mechanism [7].

Appendix A: Elastic Wave Propagation—Some Comments

The usual wave equation for modeling an isotropic, non-attenuating, isotropic, linearly whole R^3 space is given by the complete anisotropic wave equation

$$\frac{\partial^2}{\partial t^2}U^i(x,t) = \Lambda_{ijk\ell}(x)\frac{\partial^2}{\partial x^j\partial x^\ell}U^k(x,t) + D_{ik\ell}(x)\frac{\partial}{\partial x^\ell}U^k(x,t) + f_i(x,t) \tag{A.1}$$

where the spatially variable anisotropic elastic tensors $\Lambda_{ijk\ell}$ and $D_{ik\ell}$ are explicitly given in terms of the medium elastic parameters, namely: the density $\rho(x)$ and the Lamé Parameters $\mu(x)$ and $\lambda(x)$

$$\Lambda_{ijk\ell}(x) = \big(\delta^{ij}\delta^{\ell k}\big)\left(\frac{\lambda}{\rho}\right)(x) + \left(\frac{\mu}{\rho}\right)(x)\big(\delta^{i\ell}\delta^{jk}\big) + \left(\frac{\mu}{\rho}\right)(x)\big(\delta^{j\ell}\delta^{ik}\big),$$

[3] This kind of wave propagation factorized form for the wave field at the WKB short-time arrival, where the anisotropic polarization vectors dynamics is factorized out from the "scalar propagationís" naturally expected, i.e.:

$$\xi_i(x)\xi_j\big(x'\big) \simeq \int_0^\infty ds(\delta_{ij}+\Phi_{ij})\big(x,x'\big),$$

with $\{\xi_k(x)\}_{k=1,2,3}$ denoting the wave polarization vectors.

$$D_{ik\ell}(x) = \left(\frac{1}{\rho}\frac{\partial\lambda(x)}{\partial x^i}\right)\left(\delta^{\ell k}\right) + \left(\frac{1}{\rho}\frac{\partial\mu(x)}{\partial x^k}\right)\left(\delta^{\ell i}\right) + \left(\frac{1}{\rho}\frac{\partial\mu(x)}{\partial x^\ell}\right)\left(\delta^{ik}\right). \tag{A.2}$$

Let us make a detour on the usual approach and point out that the vectorial variable change

$$U^k(x,t) = [M]_{kq}(x)\phi^q(x,t) \tag{A.3}$$

where the matrix $[M]_{kq}$, $1 \le k \le 3$, $1 \le q \le 3$ is inversible everywhere and uniquely determined from the relationship PDF's first order matrixial system written below

$$\left(\Lambda_{irk\ell}(x)\frac{\partial}{\partial x^\ell}M_{ks}(x) + \Lambda_{ijkr}(x)\frac{\partial}{\partial x^j}M_{ks}(x) + D_{ikr}(x)M_{ks}(x)\right)\frac{\partial\phi^s}{\partial x^r}(x,t) \equiv 0. \tag{A.4}$$

This changes our full elastic wave equation (A.1) into a more suitable equation of the form without the troublesome first-order matrix valued derivative term

$$\frac{\partial^2}{\partial t^2}\phi^i(x,t) = \Omega_{ijq\ell}(x)\frac{\partial^2}{\partial x^j\partial x^\ell}\phi^q(x,t) + V(x)\phi^i(x,t) \tag{A.5}$$

where the new effective anisotropic tensor is now explicitly given by

$$\begin{aligned}\Omega_{ijq\ell}(x) &= \left[M^{-1}\right]_{ia}(x)\left(\Lambda_{ajk\ell}(x)\right)[M]_{kq}(x)\\ &\overset{\text{(Perturbative Hypothesis)}}{\equiv} \bar{\gamma}^{(0)}_{j\ell}(x)\delta_{ik} + g\gamma_{ijq\ell}(x)\end{aligned} \tag{A.6}$$

and the scalar potential

$$V(x) = \left[M^{-1}\right]_{ia}(x)\left\{\left(\Lambda_{ijk\ell}\left(\frac{\partial^2}{\partial x^j\partial k^\ell}M_{kq}\right)\right) + \left(D_{ik\ell}\frac{\partial}{\partial x^\ell}M_{ka}\right)\right\}(x). \tag{A.7}$$

One now can proceed as in the bulk of our Brief Report in order to produce a space-time path-integral representation for (A.5).

As a last comment with a computer oriented focus, some times it appears more suitable to write the elastic wave equation into a form of a hyperbolic system involving the propagation of a compressional wave $\theta(x,t)$ and the shear wave $U^{ij} = \frac{\partial U^i}{\partial x^j} - \frac{\partial U^j}{\partial x^i}$, namely: we have the set

$$\begin{aligned}\rho\theta_{tt} &= \big[(\Delta\lambda)\theta + 2\partial_j\lambda\partial_j\theta + \lambda\Delta\theta + (\partial_j\partial_i\mu)U^{ij}\\ &\quad + (\partial_i\mu)\left(\partial_i\theta + \Delta U^i\right) + (\partial_j\mu)(\Delta U_j + \partial_j\theta) + 2\mu\Delta\theta\big]\\ &\quad + \delta(x-\bar{x})\delta(t-\bar{t}),\end{aligned} \tag{A.8}$$

$$U^i(x,t) = -\int_{-\infty}^{+\infty}\frac{(\partial_i U^{ij} + \frac{\partial}{\partial x^j}\theta)(x',t)d^3x'}{4\pi|x-x'|}, \tag{A.9}$$

$$\rho U^i_{tt} = \mu\Delta U^i + \lambda\partial_i\theta + \mu\partial_i\theta + \partial_i\lambda\theta + \partial_j\mu\partial_i U^j + \hat{f}^i \tag{A.10}$$

where the source $\hat{f}^i$ into (A.10) must be compatible with the compressional explosive behavior delta function source in (A.8). Namely

$$\hat{f}^i = \left(\frac{1}{3}\left[\left(\prod_{\substack{j=1\\ j\neq i}}^{3} \delta(x_j - \bar{x}_j)\right)\theta(x_i - \bar{x}_i)\right] e_i\right)\delta\big(t - t'\big). \tag{A.11}$$

Numerical Analysis of (A.8)/(A.10) may be implementable by putting the medium in a Box and considering truncated Fourier expansion for all spatially variable objects involved and leading thus to a set of time-dependent ODE's for the dynamical objects Fourier coefficients.

For a compact volume Ω and the spectral eigenvalue and eigenfunctions of the Laplacian Operator in Ω (with Dirichlet conditions) and $n \in \mathbb{Z}^+$ fixed

$$\Delta\Phi_n(x) = \lambda_n \phi_n(x), \tag{A.12}$$

$$\phi_n|_{\partial\Omega} \equiv 0, \tag{A.13}$$

$$\theta^{(M)}(x,t) = \sum_{|\lambda_n|\le M} \theta_n(t)\phi_n(x), \tag{A.14}$$

$$\vec{U}^{(M)}(x,t) = \sum_{|\lambda_n|\le M} \vec{U}_n(t)\phi_n(x), \tag{A.15}$$

$$\lambda^{(M)}(x) = \sum_{|\lambda_n|\le M} \lambda_n \phi_n(x), \tag{A.16}$$

$$\mu^{(M)}(x) = \sum_{|\lambda_n|\le M} \mu_n \phi_n(x), \tag{A.17}$$

$$\delta^{(3)}_{(M)}(x - \bar{x})\delta\big(t - t'\big) = \sum_{|\lambda_n|\le M} \big(\phi_n(x)\phi_n(\bar{x})\big) \times \left[\sum_{p=-M}^{+M} \exp\big(2\pi i p\big(t - t'\big)\big)\right]. \tag{A.18}$$

Appendix B: Feynman Short-Time Amplitude

In this appendix we present our proposal proper-time Feynman asymptotic expansion for the anisotropic wave operator (3).[4] We have thus the chains of asymptotic equalities (still mathematically formal, unless the operator $\mathcal{L}$ is essentially self-adjoint and acting on a spectral—finite spectral range parameter—sub-space of $\mathcal{L}$, which means that $\mathcal{L}$ is a bounded operator there!):

$$\begin{aligned}
&\lim_{S\to 0^+} \big\langle (x,t,i)\big|\exp(is\mathcal{L})\big|\big(x',t',k\big)\big\rangle \\
&\stackrel{\text{def}}{\equiv} I\big((x,t,i);\big(x',t',k\big)\big) \\
&\overset{S\to 0^+}{\sim} \int d^N p\, dw \big(\exp\big\{is\big[-\big(w^2\big)\delta_{pq} + C_{pjq\ell}(x)p_j p_\ell + i\nu(x)w\big]\big\}\big)_{ik}
\end{aligned}$$

[4] With $\nu(x,t) \equiv 0$ for simplicity.

$$
\begin{aligned}
&\times \exp\big(ip(x-x')\big) \times \exp\big(iw(t-t')\big) \\
&\overset{S\to 0^+}{\sim} \int d^N p\, dw \big(\exp\{is[-(w^2)\delta_{pq} + C_0^2(x)p^2]\}\big)_{i\overline{n}} \\
&\times \big(\delta_{\overline{n}k} + is\varepsilon\gamma_{\overline{n}ukv}(x)p_u p_v\big).
\end{aligned}
$$

We have thus (see Refs. [1, 2], Chap. 24, (24.11) and (24.16))

$$
\begin{aligned}
I\big((x,t,i);(x',t',k)\big) &\overset{S\to 0^+}{\cong} \bigg\{\big(C^2(x)\big)^{-N/2}\bigg(\exp\bigg(-\frac{i}{2s}\frac{(x-x')^2}{C_0^2(x)}\bigg)\bigg)\delta_{i\overline{n}} \\
&\quad \times \bigg[\delta_{\overline{n}k} + is\varepsilon\delta_{\overline{n}ukv}(x)\bigg[4\bigg(\frac{i}{2s}\bigg)^2\bigg(\frac{1}{C_0^2(x)}\bigg)^2(x-x')_u(x-x')_v \\
&\quad + \bigg(\frac{i}{2s}\bigg(\frac{2}{C_0^2(x)}\bigg)\delta_{uv}\bigg)\bigg]\bigg]\bigg\} \\
&\overset{S\to 0^+}{\cong} \big(C^2(x)\big)^{-N/2}\exp\bigg(\frac{-i(x-x')^2}{2sC_0^2(x)}\bigg) \\
&\quad \times \bigg[\delta_{ik} - is\varepsilon \times \frac{1}{C_0^4(x)}\delta_{iukv}(x)\bigg(\frac{(x-x')}{s}\bigg)_u\bigg(\frac{(x-x')}{s}\bigg)_v\bigg].
\end{aligned}
\tag{B.1}
$$

Note that we have used the wearily smallness anisotropic perturbation condition to arrive at the above written final asymptotic short-times result (B.1)

$$
\delta_{\overline{n}k} - \overbrace{\bigg(\frac{\varepsilon\gamma_{\overline{n}ukv}(x)}{C_0^2(x)}\bigg)(\delta_{uv})}^{\ll\varepsilon} \sim \delta_{\overline{n}k}. \tag{B.2}
$$

Appendix C: A Toy Model for Tomographic Seismic Inversion in Scalar Path Integral Wave Propagation-Highlights

Let us consider for simplicity of our exposition scalar waves of a definite frequency $\overline{w}$ (see Chap. 20, [1, 2] for the notation details) in a medium with a spatially slowly varying index of refraction $\eta(x)$ in R^3. Namely

$$
\bigg(-\frac{\overline{w}^2}{C^2} - \frac{\overline{w}^2}{C^2B^2}\frac{1}{(\eta(x))^2}\bigg)\Delta_x U(x,\overline{w}) = 0 \tag{C.1}
$$

where we have introduced the L. Schulman somewhat artificial scale $B > 0$.

Let us further consider the "perturbative decomposition" of the medium material properties (with $\varepsilon \ll 1$)

$$
\frac{1}{(n(x))^2} = \overbrace{\frac{1}{(n_0(x))^2}}^{\text{reference}} + \varepsilon\overbrace{\frac{1}{(n_1(x))^2}}^{\text{perturbation}} \tag{C.2}
$$

where one supposes that all wave dynamics relative to the reference unperturbed medium $n_0(x)$ is known exactly.

The scattered field by the medium $n_1(x)$ is given in the lowest ε order by the following path integral representation

$$U_w^{SL}\big(x_s, x_r, [n(x)]\big) \overset{\varepsilon\to 0^+}{\cong} i\left\{\int_0^\infty ds e^{i\frac{\overline{w}^2}{C^2}S}\left\{\int_{r(0)=x_S}^{r(s)=x_r} D^F[r(\sigma)] \right.\right.$$
$$\times \exp\left(\frac{i}{4}\left(\frac{C^2B^2}{\overline{w}^2}\right)\int_0^S \left(\dot{r}^2(\sigma)n_0^2(r(\sigma))\right)\right)$$
$$\left.\left.\times\left(-\frac{1}{4}\frac{\varepsilon C^2B^2}{\overline{w}^2}\left(\int_0^S \frac{\dot{r}^2(\overline{\sigma})n_0^4(r(\overline{\sigma}))}{n_1^2(r(\overline{\sigma}))}d\overline{\sigma}\right)\right)\right\}\right\}. \quad \text{(C.3)}$$

The problem of determining the perturbing refraction index $n_1(x)$—after "Sampling" the scattered field $U_w^{SC}(x_s, x_r, [n])$ at $(2k+1)$ points $x_s = (x_1^{(s)}, \ldots, x_{2k+1}^{(s)})$, $x_r = (x_1^{(r)}, \ldots x_{2k+1}^{(r)})$, namely: One knows the set

$$\left\{U_w^{SC}(x_p^{(s)}, x_q^{(s)})\right\}_{\substack{1\le p\le 2k+1\\ 1\le q\le 2k+1}}$$

at the source $x_q^{(s)}$ and receiver $x_p^{(r)}$ grid point of our Geophysical surveillance, lead us to the seismic tomographic scalar inversion problem in the path integral framework in order to estimate $\eta_1(s)$.

We, thus, consider firstly the Fourier Transformed of the refraction perturbing index (its square inverse for sure!)

$$\left(\frac{1}{n_1^2(\vec{r})}\right) = \frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\vec{k}\vec{r}} \times \widetilde{V}(\vec{k}). \quad \text{(C.4)}$$

After considering the sampling of $\widetilde{V}(\vec{k})$ at $N = 2k(k+1)$ points; $\{\widetilde{V}(k_{-N}), \ldots, \widetilde{V}(k_N)\}$ still to be determined from the linear equation (C.3) namely $(2k+1)^2 = n^2 = 2N+1 = (2\cdot 2k(k+1))+1)$, one obtains the short-time "seismological data recorded equations" [7].

$$\overbrace{\begin{bmatrix} A_{11}(K_{-N}) & \cdots & A_{11}(K_N)\\ A_{12}(K_{-N}) & \cdots & A_{17}(K_N)\\ \cdots & \cdots & \cdots\\ A_{2k+1,2k+1}(K_{-N}) & \cdots & A_{2k+1,2k+1}(K_N)\end{bmatrix}}^{[M]_{2N+1,2N+1}} \overbrace{\begin{bmatrix}\widetilde{V}(K_{-N})\\ \vdots\\ \widetilde{V}(K_N)\end{bmatrix}}^{[\vec{\widetilde{V}}]} = \overbrace{\begin{bmatrix} U^{SC}(x_1^{(r)}, x_1^{(s)})\\ \vdots\\ U^{SL}(x_p^{(r)}, x_q^{(s)})\\ \vdots\\ U^{SL}(x_{2k+1}^{(r)}, x_{2k+1}^{(s)})\end{bmatrix}}^{[\vec{U}^{SL}]} \quad \text{(C.5)}$$

with the exactly evaluated path integral coefficients sources-receivers propagations as given below by (C.6). (One can evaluate exactly them for instance by means of the short-proper time limit $B\to\infty$ in Schulman's approach, however by keeping $\lim_{\varepsilon\to 0^+}[\varepsilon(\frac{C^2B^2}{\overline{w}^2})] < +\infty$.)

$$A_{pq}(\vec{k}_j) = -\frac{1}{4}\varepsilon\left(\frac{C^2B^2}{\overline{w}^2}\right)\times\left\{\int_0^\infty ds\, e^{i\frac{\overline{w}^2}{C^2}S}\right.$$
$$\times\left[\int_0^S d\overline{\sigma}\left(\int_{\vec{r}(0)=\vec{x}_q^{(s)}}^{\vec{r}(s)=\vec{x}_p^{(r)}} D^F[\vec{r}(\sigma)]\right.\right.$$

$$\times \exp\left(-\frac{i}{4}\left(\frac{C^2B^2}{\overline{w}^2}\right)\int_0^2 \vec{\dot{r}}^2(\sigma) \times \left(n_0(\vec{r}(\sigma))\right)^2 d\sigma\right)$$

$$\times \exp\left(i\vec{k}_j \cdot \vec{r}(\overline{\sigma})\right) \times \left(\vec{\dot{r}}^2(\overline{\sigma})\right) \times \left(n_0^4(\vec{r}(\sigma))\right)\Big)\Big]\Big\}. \tag{C.6}$$

Note that the well-posed solution of the linear matrix equation (C.5) is given by the standard methods of Linear Algebra

$$[\vec{\tilde{V}}] = \left([M]^T[M]\right)^{-1}\left([M]^T \cdot \left[\vec{U}^{SL}\right]\right). \tag{C.7}$$

After evaluating the FFT coefficients $\{\widetilde{V}(\vec{k}_{-j})\}_{N\leq j\leq N}$, one can "estimate" (C.4).

Finally the problem of handling Caustics and WkB next corrections with a non-trivial determinant associated to the density of paths, can be in principle straightforward implemented in our proposed tomographic path integral scheme by just considering one-loop corrections to the path integral source-receiver propagator (C.6), namely, the next one-loop correction leading to the Van-Vleck-Morrete determinant is coming from the following quadratic path integral

$$\int_{\vec{r}(0)=\vec{x}_q^{(s)}}^{\vec{r}(s)=\vec{x}_p^{(r)}} D^F\left[\vec{r}(\sigma)\right] e^{\left(\frac{i}{4}\left(\frac{C^2B^2}{\overline{w}^2}\right)\int_0^S [\vec{r}^2_{CL}(\sigma)+\vec{r}^2_f(\sigma)](n_0(\vec{r}_{CL}(\sigma)))^2 d\sigma\right)}$$
$$\times \left(e^{i\vec{k}_j(\vec{r}_{CL}(\overline{\sigma})+\vec{r}_f(\overline{\sigma}))}\right)$$
$$\times \left[\left(\vec{\dot{r}}_{CL}(\overline{\sigma}) + \vec{\dot{r}}_f(\overline{\sigma})\right)^2 \times \eta_0^4\left(\vec{r}_{CL}(\overline{\sigma})\right)\right] \tag{C.8}$$

where the geometrical ray path classical trajectory is given by the well know ray equation

$$\begin{cases} \frac{d}{d\sigma}(\vec{\dot{r}}^{CL}(\sigma)\eta_0^2(\vec{r}^{CL}(\sigma)) = 0, \\ \vec{\dot{r}}^{CL}(0) = \vec{x}_q^{(s)}; \qquad \vec{\dot{r}}^{CL}(s) = \vec{x}_p^{(r)}. \end{cases} \tag{C.9}$$

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